INFORMATION AND CONTROL 1, 165–169 (1958)

# A Note on the Noise-Widened Oscillator Spectrum

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Random noise in an oscillator results in the widening of the generated spectral line. When thermal noise only comes into the picture, one arrives at the formula:

$$\frac{\delta\omega_0}{\omega_0} = \frac{nkT\omega_0}{8\pi W_i Q^2}$$

LIST OF SYMBOLS

| | |
|---|---|
| $\omega_0$ | Angular frequency of oscillation |
| $\delta\omega_0$ | Width of spectral line |
| $k$ | Boltzmann's constant ($k = 1.38\ 10^{-23}$ joule) |
| $T$ | Temperature in degrees Kelvin |
| $W_i$ | Power at amplifier input terminals |
| $Q$ | Quality factor of circuits when there is only one tank circuit |
| $n$ | Noise figure of the amplifier (in power ratio) |
| $t$ | Time |
| $x$ | Amplitude of oscillations |
| $f(x)$ | Describing function of non-linear part, |
| $\mathcal{C}(j\omega)$ | Transfer function of linear part |
| $B$ | Bandwidth |
| $z$ | Disturbance, |
| $a$ | Amplitude of disturbance |
| $\varphi$ | Outphasing of linear network |
| $W$ | Power of noise |
| $\bar{b}^2$ | Mean square of noise voltage, |
| $R$ | Input resistance of amplifier |

## I. INTRODUCTION

There is, strictly speaking, no such thing in nature as a monochromatic oscillator; it is well known that random noise is ultimately responsible for the widening of emitted spectral lines. Once all kinds of spurious signals, such as microphonics, etc., have been eliminated, there still remains thermal noise (whose power spectrum is given by the Nyquist equation). So, it is of interest to compute the ultimate width of the transmitted spectral line, due to thermal noise. The aim of this paper is to analyze the effect of thermal noise on a conventional electronic oscillator.

## II. A NONLINEAR MODEL FOR AN OSCILLATOR

The "describing function" method has become a standard approach for the analysis of nonlinear devices, such as amplifiers. Referring to Fig. 1 the complete loop comprises two elements in cascade. The first (the amplifier) gives an output that is limited and distorted by saturation. This element is assumed to change a sinusoidal input $x \cos \omega t$ into another signal, whose first harmonic only is taken into consideration. Of course, the output includes higher harmonics, but as an essential assumption, those higher harmonics are filtered out by the following linear network.

So, for present computation, we shall simply write the output as

$$x\, f(x) \cos \omega t$$

where $f(x)$ represents the action of the amplifier, without any time feature, such as phase shifting. The second element will consist of a linear network defined by its conventional transfer function $\mathcal{C}(S)$.

The oscillation takes place for any pair of values $x$ and $\omega$ such that

$$f(x)\, \mathcal{C}(j\omega) = 1, \underline{|\pi} = -1 \tag{1}$$

This simple equation (1) gives an account of some causes of instability relating to the operation of the oscillator, independently from alterations of the linear network.

If $f(x)$ does not bring any phase-shift into the picture, the actual frequency $\omega_0$ of the oscillation is given by the necessary and sufficient relationship:

$$\text{Phase of } \mathcal{C}(j\omega_0) = \pi \tag{2}$$

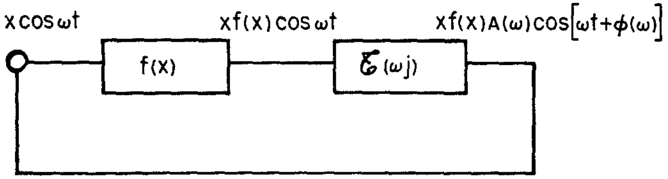


Fig. 1. Undisturbed oscillator.

The amplitude $x$ automatically adjusts itself at a value $x_0$ such that

$$f(x_0) \mid \mathcal{C}(j\omega_0) \mid = 1 \tag{3}$$

In this case, we must note that any drift in the gain of the amplifier cannot result in any frequency drift. In practical cases $f(x)$ is not completely free from time features (for instance, in high frequency oscillators, the finite velocity of electrons causes a phase delay closely related to the signal amplitude). But, in this paper, we are interested in the *fundamental* limitation of the accuracy of an oscillator due to noise, and we need not consider any simple amplitude effect that can be eliminated (theoretically at least).

Thermal noise is not only characterized by its amplitude: it has a spectral distribution, given by Nyquist's relationship,

$$W_N = kTB \tag{4}$$

where $W_N$ is the power generated in a resistive dipole at absolute temperature $T$, inside a bandwidth $B$. ($k$ is Boltzmann's constant: $k = 1.38$ $10^{-23}$ joule.) The following computation gives the effect of the noise (whose spectrum is known) on the width of an emitted spectral line.

## III. PULLING OF OSCILLATORS

The phenomenon of frequency pulling has often been described. We will use the approach of Cahen and Loeb (1952). In Fig. 2, the conventional representation of the oscillator includes a new input, at the differential element $D$.

Let us suppose that, at this input, we apply a voltage

$$z = a \cos (\omega_0 + \delta\omega)t$$

and let $\omega_0$ be the angular frequency of the oscillation and $x_0$ the amplitude of the oscillation when $a = 0$. When $\delta\omega$ is very large, there is a beat at frequency $\delta\omega$ but as $\delta\omega$ decreases, synchronization takes place.

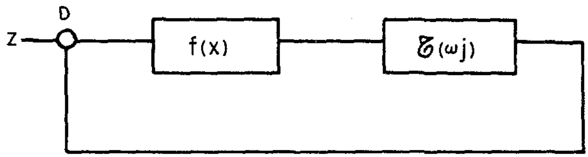


FIG. 2. Disturbed oscillator.

The largest $\delta\omega$ that allows synchronization is $\delta\omega_0$, given by:

$$\delta\omega_0 = \frac{a}{x_0} \Big/ \frac{\partial\varphi(\omega_0)}{\partial\omega} \tag{5}$$

In this equation $\varphi$ is the phase-shift given by the linear network. In this formula, one has to assume that $\varphi$ does not depend on $x$.

The general formula is somewhat more complicated. When the linear circuit is a resonator whose natural frequency is $\omega_0$ and quality factor is $Q$, we get:

$$\frac{\delta\omega_0}{\omega_0} = \frac{a}{x_0} \cdot \frac{1}{2Q} \tag{6}$$

Now we will compute what happens when instead of a sine input $z = \cos(\omega_0 + \delta\omega)t$, we have a thermal noise.

## IV. EFFECT OF NOISE

A subtle point of the theory is this: we consider a *still unknown* bandwidth of the spectral line $\delta\omega_0$. The noise power at the amplifier input terminals is

$$W = \frac{nkT\delta\omega_0}{2\pi} \tag{7}$$

where $n$ is the noise figure (power ratio) of the amplifier. If this noise figure is 6 db, we get $n = 4$. We have written $2\pi$ as the denominator assuming that the noise power is equally divided between the source and the amplifier. If now $R$ is the input resistance of the amplifier, we can compute the mean square $\bar{b}^2$ of the noise voltage

$$\bar{b}^2 = \frac{nkTR\delta\omega_0}{2\pi} \tag{8}$$

This mean square voltage can be considered as resulting from the following random process: a sine voltage $a \cos \omega' t$ is applied at the input of the amplifier, where $\omega'$ is a random frequency lying between $\omega_0$ and $\omega_0 + \delta\omega_0$ with uniform probability. The power of this random process is equal to $W$ in Eq. (7) if

$$a^2 = 2\bar{b}^2 \tag{9}$$

and

$$a = \sqrt{\frac{nkTR\delta\omega_0}{\pi}} \tag{10}$$

The spectral line width $\delta\omega_0$ will be known by putting the value of $a$ of (10) into Eq. (6).

$$\frac{\delta\omega_0}{\omega_0} = \frac{nkTR\omega_0}{\pi x_0^2} \cdot \frac{1}{4Q^2} \tag{11}$$

Now $\dfrac{x_0^2}{2R}$ is the input power $W_i$ to the amplifier. Eventually:

$$\frac{\delta\omega_0}{\omega_0} = \frac{nkT\omega_0}{8\pi W_i Q^2} \tag{12}$$

## V. CONCLUSIONS

First, Eq. (12) shows how the accuracy of an oscillator is improved when $Q$ is increased, but there is no close relationship between $\delta\omega_0$ and the width of the resonance curve $\omega_0/Q$. Actually the power level $W_i$ at the input terminals of the amplifier plays a very important role. A second point is that, even if the non-linear part of the loop is perfect, i.e. does not cause any phase shift, the noise results in a widening of the emitted spectral line.

RECEIVED: April 9, 1957.

## REFERENCE

CAHEN, G., AND LOEB, J. (1952). "Entraînement des oscillateurs non-linéaires filtrés." *Ann. Télécommunications* **7** (10), 411–413.